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On the Additive Completion of Squares

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1. INTRODUCTION

Let B be a subset of integers. Let N be a sufficiently large integer. Let b denote the "generic" element of B and λ a positive integer. We assume that every $n(<N)$ can be written, in at least one way, as $n=b+\lambda^2$. Then, improving a result of Abbott [1], we prove

THEOREM 1. *We have* $\sum_{b\in B} 1 > (1.15)\sqrt{N}$.

2. PRELIMINARY LEMMAS

LEMMA 1. *If* $f(n)\geqslant 0$, *then* $\sum_{b+\lambda^2\leqslant N} f(b+\lambda^2) \geqslant \sum_{n\leqslant N} f(n)$.

Proof. Clear.

LEMMA 2. $\sum_p \sqrt{N-b} \geqslant (1+o(1))\,N$.

Proof. We take $f(n)=1$ and simplify.

LEMMA 3. *We have*

$$\sum_b (N-b)^{3/2} \leqslant \frac{3}{2} N \sum_b (N-b)^{1/2} - \left(\frac{3}{4}+o(1)\right) N^2.$$

Proof. We take $f(n)=n$, then

$$\frac{N^2}{2} \leqslant \sum_{m\leqslant N} m \leqslant \sum_b \sum_{\lambda\leqslant\sqrt{N-b}} (b+\lambda^2)$$
$$\leqslant \sum_b \left\{ b\sqrt{N-b} + \frac{(N-b)^{3/2}}{3} \right\} + o(N^2).$$

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$$=\sum_b \sqrt{N-b}\left(N-\frac{2}{3}(N-b)\right)+o(N^2)$$

$$=N\sum_b \sqrt{N-b}-\frac{2}{3}\sum_b (N-b)^{3/2}+o(N^2)$$

and the lemma follows.

3. Proof of Theorem 1

By Holder's inequality,

$$\left(\sum \sqrt{(N-b)}\right)^3 < \left(\sum_b (N-b)^{3/2}\right)\left(\sum_b 1\right)^2.$$

Hence using Lemma 3,

$$\left(\sum_b 1\right)^2 \geqslant \left(\sum_b \sqrt{N-b}\right)^3 \left(\left(\frac{3}{2}N\sum_b (N-b)^{1/2}\right)-\left(\frac{3}{4}+o(1)\,N^2\right)\right)^{-1}.$$

The right side is, thanks to Lemma 2, bigger than $(\frac{4}{3}+o(1))\,N$. This proves the theorem.

4. Generalisations

The above procedure could be generalised. Let, as before, B be a set with "generic" element b and λ an integer. Let $f(x)$ be a polynomial with non-negative integer coefficients and with leading term $c>0$. Assume that every $n(<N)$ can be written as $b+f(\lambda)$ in at least one way. Then we can prove, in a similar way,

Theorem 2. *We have*

$$\sum_{b\in B} 1 > \left(2c-\frac{2}{r+1}+o(1)\right)^{1/r} N.$$

Remark. (1) This improves the lower bound $(1+(n-1)/2n^2+o(1))\,N$ obtained by Donagi and Herzog [2].

(2) In the special case $f(\lambda)=\lambda^3$, we get the lower bound $(1.5)^{1/3}$ which improves 1.137 obtained by Abbott [1].

REFERENCES

1. H. L. ABBOTT, On the additive completion of sets of integers, *J. Number Theory* **17** (1983), 135–143.
2. R. DONAGI AND M. HERZOG, On the additive completion of polynomial set of integers, *J. Number Theory* **3** (1971), 150–154.